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Derivative in Rings with Ideals

Owino J. Jagongo*, Kewamoi C. Sogomo, Moses N. Gichuki, Marion N. Nyongesa

Department of Mathematics, Egerton University, Egerton, Kenya

Correspondence author: jobjagongo92@gmail.com

Abstract -A ring is a set that has two binary operations of addition and multiplication. Moreover, for a set to be a ring it must be a group with respect to addition and multiplication should be distributive over addition. A ring homomorphism is a function that exists between two rings and respects the structure. In this paper, our aim is to define a derivative on rings with ideals and prove certain properties of the derivative on rings with ideals. The results obtained in this paper will be vital in obtaining optimization results in rings with ideals.

Index Terms -Rings; Ideals; Ring Homomorphism; Derivative

1. INTRODUCTION

In mathematics, a derivative can be defined to be the rate at which one object or quantity changes with respect to another. Geometrically, a derivative is defined to the slope of a curve at a point on that particular curve. The main problem that seems to make the study of a derivative in rings with ideals is the fact that it is algebraic in nature and yet the derivative is mostly studied in classical analysis [4]. With the introduction of infinitesimals by [6] and [7], it has been possible to study some parts of analysis in such toposes. The concept of infinitesimals can also be used to define the derivative in rings with ideals. This paper thus aims at defining a derivative in rings with ideals and then proving certain properties of the derivative on rings with ideals.

2. LITERATURE REVIEW

2.1 Ideal

An ideal is defined as a subset I in a ring R that forms an additive group and has the characterization that that whenever an element x belongs to the ring R and the element y belongs to the subset I of R then xy and yx belong to the subset I of R [3] [5]. Infinitesimals in R/∞ forms an ideal.

2.2 Ring and Ring Homomorphism

A ring is a set that has two binary operations of addition multiplication such that multiplication is distributive over addition and the set is a group with respect to addition [2]. The set of infinitesimals form a ring. Moreover the set of R/∞ forms a ring [9].

A function f is said to be a ring homomorphism if $\forall a, b \in R$ then the following are fulfilled:

i. $f(a+b) = f(a) + f(b)$
ii. $f(ab) = f(a)f(b)$

If e is the identity of a ring with respect to addition then $x = e + x$ [1]. This shows that if G is a group with respect to addition and e is the identity then $f(e)$ is the identity in $f(G,+)$. The identity of a group with respect to multiplication is thus defined.

3. MAIN RESULTS

3.1 Definition

If f is a ring homomorphism, and

$$f(x+d) - f(x) = Ad + d$$

Then A can be called a derivative. $x, d \in R$ Where $d \in ideal$ of R.

If g is a ring homomorphism then:

$$ring \times ideal = ideal$$

$$g(ring \times ideal) = g(ring)g(ideal) = g(ideal)$$

Therefore, $g : ideal \to ideal$

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Hence A is a derivative of f at x.

3.1.1 Consequence of the definition

Suppose that f is a constant, that is, for some $c \in R$ we have $f(x) = c \forall x$ then,

$$f(x+d) = c = f(x) + Ad + d$$

But $Ad = 0$

Thus,

$$f(x+d) = c = f(x) + d.0 + d = f(x) + d$$

3.2 Derivative properties

3.2.1 Theorem 1

If A is a derivative then $A(f.g) = f.A(g) + A(f).g$

Proof

Following Mac Larty (1992)we show that $A(f.g) = f.A(g) + A(f).g$

Indeed

$$A(f.g) = f(x+d)g(x+d) - f(x)g(x)$$

$$= f(x+d)g(x+d) - f(x+d)g(x) + f(x+d)g(x) - f(x)g(x)$$

$$= f(x+d)[g(x+d) - g(x)] + [f(x+d) - f(x)]g(x)$$
$$= f(x+d)[Ad+d] + [Ad+d]g(x)$$
$$= f(x)Ad + f(x)d + f(d)Ad + f(d)d + Adg(x) + dg(x)$$
$$= fA_g d + d + 0 + 0 + gA_f d + d$$

$$= d(fA_g + gA_f) + d$$

$$= fA_g + gA_f$$

Hence the proof.

3.2.2 Theorem 2

If A is a derivative and suppose f and g are ring homomorphisms which are differentiable then

$$A(f+g) = Af + Ag$$

Proof

Indeed,

$$A(f+g) = [f(x+d) + g(x+d)] - [f(x) + g(x)]$$

$$= [f(x+d) - f(x)] - [g(x+d) - g(x)]$$
$$= [A_f d + d] + [A_g d + d]$$
$$= Af + Ag$$

Hence the proof.

3.2.3 Theorem 3

If A is a derivative and suppose that f is a homomorphism and c is a constant then,

$$A(cf) = cAf$$

Proof

$$A(cf) = c[f(x+d) - f(x)]$$

$$= c[A_f d + d]$$
$$= cAf$$

Hence the proof.

3.2.4 Theorem 4

If A is a derivative and f, g are homomorphisms in a ring R, then

$$A(\frac{f}{g}) = \frac{gAf - fAg}{g^2}$$

Proof

$$A(\frac{f}{g}) = \frac{f(x+d)}{g(x+d)} - \frac{f(x)}{g(x)}$$

$$= \frac{g(x)f(x+d) - g(x+d)f(x)}{g(x+d)g(x)}$$

$$= \frac{g(x)f(x+d) - g(x)f(x) + g(x)f(x) - g(x+d)f(x)}{g(x+d)g(x)}$$

$$= \frac{g(x)[f(x+d) - f(x)] - f(x)[g(x+d) - g(x)]}{g(x+d)g(x)}$$

$$= \frac{g(x)[A_f d + d] - f(x)[A_g d + d]}{g(x+d)g(x)}$$

$$= \frac{(gA_f)d - (fA_g)d}{g^2}$$

$$= \frac{gAf - fAg}{g^2}$$

Hence the proof.

Discussion and Conclusion

[8] showed that a ring with infinitesimals is a topos and hence derivatives can be defined in a topos. This paper has shown that Sukhinin type of a derivative is also definable in such rings hence in a topos. It should then be possible to obtain optimization results in such a topos in a similar was as in [10].

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